Singapore International Mathematics Olympiad
National Team Training

16 March 2002

1 Let S be the set of all binary sequences of length n:

$$S = \{\underbrace{0...0}_{n}, \underbrace{0...0}_{n-1}1, ..., \underbrace{1...1}_{n}\}.$$

Let the number of places in which two sequences differ be called the *distance*, $\mathcal{D}$ between them, and suppose that $T_{\mathcal{D}} \subseteq S$ is such that the distance between any two members of $T_{\mathcal{D}}$ is at least $\mathcal{D}$.

Show that, when $\mathcal{D}$ is odd, $\mathcal{F}(n, \mathcal{D})$, the maximum possible number of elements of $T_{\mathcal{D}}$, satisfies the following inequality:

$$\mathcal{F}(n, \mathcal{D}) \leq \left\lfloor \frac{2^n}{\sum_{i=0}^{\frac{\mathcal{D}-1}{2}} \binom{n}{i}} \right\rfloor.$$

Construct such a subset for each of the following cases:

(a) $n = 5, \mathcal{D} = 3$,

(b) $n = 7, \mathcal{D} = 3$.

2 Let m, n be natural numbers with $m \geq n \geq 2$. Show that the number of polynomials of degree $2n - 1$ with distinct coefficients from the set $\{1, 2, ..., 2m\}$ which are divisible by $x^{n-1} + \cdots + x + 1$ is

$$2^n n! \left(4\binom{m+1}{n+1} - 3\binom{m}{n}\right).$$

3 The benches of the Great Hall of the Parliament of Never Enough are arranged in a rectangle of 10 rows of 10 seats each. All the 100 MPs have different salaries. Each of them asks all his neighbours (sitting next to, in front of, or behind him, i.e. 4 members at most) how much they earn. They feel a lot of envy towards each other; an MP is content with his salary only if he has at most one neighbour who earns more than himself. What is the maximum possible number of MPs who are satisfied with their salaries?

4 Determine all integers $n > 1$ such that for all $a_1, a_2, ..., a_n > 0$

$$\sqrt[n-1]{\frac{\sum_{i=1}^{n} \prod_{j \neq i} a_j}{n}} \leq \sqrt{\frac{2\sum_{1 \leq i < j \leq n} a_i a_j}{n(n-1)}}.$$

Determine all instances of equality.

5 Given any positive integer n, can the polynomial

$$f(x) = (x^2 + x)^{2^n} + 1$$

be written as the product of two nonconstant polynomials with

(a) integral coefficients?

(b) rational coefficients?