Singapore International Mathematics Olympiad
National Team Training

25-05-02

1 (UK) Some three nonnegative real numbers p, q, r satisfy

$$p + q + r = 1.$$

Prove that

$$7(pq + qr + rp) \leq 2 + 9pqr.$$

2 (Austrian-Polish Mathematics Competition) For all real numbers $a, b, c \geq 0$ such that $a + b + c = 1$, prove that

$$2 \leq (1 - a^2)^2 + (1 - b^2)^2 + (1 - c^2)^2 \leq (1 + a)(1 + b)(1 + c)$$

and determine when equality occurs for each of the two inequalities.

3 (Iran) Let ABC be a triangle with $BC > CA > AB$. Choose points D on BC and E on BA such that

$$BD = BE = AC.$$

The circumcircle of triangle BED intersects AC at P and the line BP intersects the circumcircle of triangle ABC again at Q. Prove that $AQ + QC = BP$.

4 (Modified from 6th IMC) Let S be the set of finite words of the letters x, y, z. A word can be transformed into an other word by the following two operations:

(a) We choose any part of the word and replicate it, for example $y\underline{yzx}z \to y\underline{yzxyzx}z$;

(b) (The reverse of the first operation) If two consecutive parts of the word are identical, we may omit one of them, for example $\underline{xyzxyz}yx \to \underline{xyz}yz$.

Show that any word can be transformed into a word of length 8.

5 (Poland) In a convex hexagon $ABCDEF$, $\angle A + \angle C + \angle E = 360^0$ and

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.$$

Prove that $AB \cdot FD \cdot EC = BF \cdot DE \cdot CA$.