Singapore International Mathematics Olympiad
National Team Training

25-05-02

1 Idea: To 'denormalise'. Or use Lagrange Multipliers.

2 Similar to Q1. Or use Lagrange Multipliers.

3 (Iran) Let ABC be a triangle with $BC > CA > AB$. Choose points D on BC and E on BA such that

$$BD = BE = AC.$$

The circumcircle of triangle BED intersects AC at P and the line BP intersects the circumcircle of triangle ABC again at Q. Prove that $AQ + QC = BP$.

Solution

Let Q' be the point on line BP such that $\angle BEQ' = \angle DEP$. Then

$$\angle Q'EP = \angle AED - \angle BEQ' + \angle DEP = \angle BED.$$

Since $BE = BD$, $\angle BED = \angle EDB$, and because $BEPD$ is cyclic, $\angle EDB = \angle EPB$.
Therefore, $\angle Q'EP = \angle EPB = \angle EPQ'$ and $Q'P = Q'E$.
Since $BEPD$ and $BAQC$ are cyclic, we have

$$\angle BEQ' = \angle DEP = \angle DBP = \angle CAQ,$$

$$\angle Q'BE = \angle QBA = \angle QCA.$$

Combining this with $BE = AC$, we have triangles EBQ' and ACQ are congruent. Thus $BQ' = QC$ and $EQ' = AQ$. Therefore,

$$AQ + QC = EQ' + BQ' = PQ' + BQ',$$

which equals BP if Q' is between B and P.
E is on $\vec{BA}$ and P is on $\vec{CA}$, so E and P are on the same side of $\overline{BC}$ and thus $\overline{BD}$.
D is on $\vec{BC}$ and P is on $\vec{AC}$, so D and P are on the same side of $\overline{BA}$ and thus $\overline{BE}$.
Thus $BEPD$ is cyclic in that order and we have

$$\angle BEQ' = \angle DEP < \angle BEP.$$

It follows that Q' lies on segment BP, as desired.

4 (Modified from 6th IMC) Let S be the set of finite words of the letters x, y, z. A word can be transformed into an other word by the following two operations:

(a) We choose any part of the word and replicate it, for example $y\underline{yzx}z \to y\underline{yzxyzx}z$;

(b) (The reverse of the first operation) If two consecutive parts of the word are identical, we may omit one of them, for example $\underline{xyzxyz}yx \to \underline{xyz}yz$.

Show that any word can be transformed into a word of length 8.

Solution

Let us first define an equivalence relation $\sim$ on the set S, where $u \sim v$, iff the words u, v can be obtained from one another by suitably applying a combination of the two operations.

Then let us prove the following lemma: If a word $u \in S$ contains at least one of each letter, and $v \in S$ is an arbitrary word, then there exists a word $w \in S$ such that $uvw \sim u$.

If v contains a single letter, say x, then write u in the form u_1xu_2 and choose $w = u_2$. Then $uxw = (u_1xu_2)xu_2 = u_1(xu_2xu_2) \sim u_1xu_2 = u$, as desired. Note that if $v = u_1x$, then we can choose $w = x$.

In the general case, let the letters of v be $a_1, a_2, ..., a_k$. Then there exists some words $w_1, w_2, .., w_k \in S$ such that $(ua_1)w_1 \sim u$, $(ua_1a_2)w_2 \sim ua_1$, ..., $(ua_1a_2...a_k)w_k \sim ua_1a_2...a_{k-1}$. Then

$$u \sim ua_1w_1 \sim ua_1a_2w_2w_1 \sim ... \sim ua_1a_2...a_kw_k...w_2w_1 = uv(w_k...w_2w_1),$$

so $w = w_k...w_2w_1$ is a good choice.

Consider now an arbitrary word a, which contains at least 9 words. We shall prove that there exists a shorter word equivalent to a. If a word can be written in the form $uvvw$, we can shorten it by $uvvw \sim uvw$, and thus we can assume a to be not of this form.

Write $a = bcd$, where b and d are the first and last four letters of a respectively. We shall prove that $a \sim bd$. It is easy to see that if b or d contains only 1 or 2 letters, then they can be reduced. So b and d contains all three letters x, y, z. Then by the lemma there exists a word e such that $b(cd)e \sim b$ and a word f such that $def \sim d$, and thus

$$a = bcd \sim bc(def) \sim bc(dedef) = (bcde)(def) \sim bd$$

as desired.

5 (Poland) In a convex hexagon $ABCDEF$, $\angle A + \angle C + \angle E = 360^0$ and

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.$$

Prove that $AB \cdot FD \cdot EC = BF \cdot DE \cdot CA$.

Solution 1

Construct point G on the exterior of the hexagon so that triangle GBC is similar to triangle FBA (and with the same orientation). Then $\angle DCG = 360^0 - (\angle GCB + \angle BCD) = \angle DEF$ and

$$\frac{GC}{CD} = \frac{FA \cdot \frac{BC}{AB}}{CD} = \frac{FE}{ED},$$

so $\triangle DCG \sim \triangle DEF$.

Now $\frac{AB}{BF} = \frac{CB}{BG}$ by similar triangles, and $\angle ABC = \angle ABF + \angle FBC = \angle CBG = \angle FBC = \angle FBG$. Thus $\triangle ABC \sim \triangle FBG$, and likewise $\triangle EDC \sim \triangle FDG$. Then

$$\frac{AB}{CA} \cdot \frac{EC}{DE} \cdot \frac{FD}{BF} = \frac{FB}{GF} \cdot \frac{FG}{DF} \cdot \frac{FD}{BF} = 1,$$

as required.

Solution 2

Position the hexagon in the complex plane and let $a = B - A, b = C - B, ..., f = A - F$. The product identity implies that $|ace| = |bdf|$ and the angle equality implies that $\frac{-b}{a} \cdot \frac{-d}{c} \cdot \frac{-f}{e}$ is real and positive. Hence $ace = -bdf$. Also $a + b + c + d + e + f = 0$. Multiplying this by ad and adding $ace + bdf = 0$ gives

$$a^2d + abd + acd + ad^2 + ade + adf + ace + bdf = 0$$

which factors to $a(d + e)(c + d) + d(a + b)(f + a) = 0$. Thus

$$|a(d + e)(c + d)| = |d(a + b)(f + a)|$$

as desired.