Singapore International Mathematics Olympiad
National Team Training

28-05-02

1 (Hungary) Determine if there exists an infinite sequence of positive integers such that

(a) no term divides any other term;

(b) every pair of terms has a common divisor greater than 1, but no integer greater than 1 divides all of the terms.

2 (Korea) Let $a_1, a_2, ..., a_{2002}$ be nonnegative real numbers satisfying the following two conditions:

(a) $a_1 + a_2 + \ldots + a_{2002} = 2$;

(b) $a_1a_2 + a_2a_3 + \ldots + a_{2002}a_1 = 1$.

Let $S = a_1^2 + a_2^2 + \ldots + a_{2002}^2$. Find the maximum and minimum possible values of S.

3 (Russia) Prove the inequality

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \leq \frac{2}{\sqrt{1+xy}}$$

for $0 \leq x, y \leq 1$.

4 (Iran) Suppose that n is a positive integer. The n-tuple $(a_1, a_2, \ldots, a_n)$ of positive integers is said to be **good** if $a_1 + a_2 + \ldots + a_n = 2n$ and if for every k between 1 and n, no k of the n integers add up to n. Find all n-tuples that are good.

5 (Turkey) Some of the vertices of the unit squares of an $n \times n$ chessboard are coloured such that any $k \times k$ square formed by these unit squares has a coloured point on at least one of its sides. If $l(n)$ denotes the minimum number of coloured points required to ensure the above condition, prove that

$$\lim_{n\to\infty} \frac{l(n)}{n^2} = \frac{2}{7}.$$

6 (China) Let ABC be an acute angle triangle with $\angle C > \angle B$. Let D be a point on the side BC such that $\angle ADB$ is obtuse, and let H be the orthocenter of triangle ABD. Suppose that F is a point inside triangle ABC and is on the circumcircle of triangle ABD. Prove that F is the orthocenter of triangle ABC if and only if both of the following are true: $HD \parallel CF$, and H is on the circumcircle of triangle ABC.

7 (Bulgaria) The vertices of A, B and C of an acute-angled triangle ABC lie on the sides B_1C_1, C_1A_1 and A_1B_1 of triangle $A_1B_1C_1$ such that $\angle ABC = \angle A_1B_1C_1$, $\angle BCA = \angle B_1C_1A_1$, and $\angle CAB = \angle C_1A_1B_1$. Prove that the orthocenters of the triangle ABC and triangle $A_1B_1C_1$ are equidistant from the circumcenter of triangle ABC.

8 (Vietnam) Let A', B', C' be the respective midpoints of the arcs BC, CA, AB, not containing points A, B, C, respectively, of the circumcircle of the triangle ABC. The sides BC, CA, AB meet the pairs of segments

$$\{C'A', A'B'\}, \{A'B', B'C'\}, \{B'C', C'A'\}$$

at the pair of points

$$\{M, N\}, \{P, Q\}, \{R, S\}$$

respectively. Prove that $MN = PQ = RS$ if and only if the triangle ABC is triangle.

9 (Original) Given any integer $m > 2$, with $m \equiv 2 \pmod 4$, show that there exist two nonconstant polynomials $g(x)$ and $h(x)$ such that the polynomial $f(x)$, given by

$$f(x) = x^4 - mx^2 + 1$$

can be written in the form $f(x) \equiv g(x)h(x) \pmod p$.

10 (Czech) Find all pairs of real numbers a and b such that the system of equations

$$\frac{x+y}{x^2+y^2} = a, \frac{x^3+y^3}{x^2+y^2} = b$$

has a solution in real numbers (x, y).