Singapore International Mathematics Olympiad
National Team Training

29-05-02

1 (Russia) Find all infinite bounded sequences $a_1, a_2, ...$ of positive integers such that for all $n > 2$,

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}, a_{n-2})}.$$

2 (Russia) Four natural numbers have the property that the square of the sum of any two of the numbers is divisible by the product of the two. Show that at least three of the four numbers are equal.

3 (Iran) Let I be the incenter of triangle ABC and let AI meet the circumcircle of ABC at D. Denote the feet of the perpendiculars from I to BD and CD by E and F, respectively. If $IE + IF = \frac{AD}{2}$, calculate $\angle BAC$.

4 (Italy) Let X be a set with $|X| = n$, and let $A_1, A_2, \ldots, A_m$ be subsets of X such that

(a) $|A_i| = 3$ for $i = 1, 2, \ldots, m$.

(b) $|A_i \cap A_j| \leq 1$ for all $i \neq j$.

Prove that there exists a subset of X with at least $\lfloor \sqrt{2n} \rfloor$ elements, which does not contain A_i for $i = 1, 2, ..., m$.

5 (Russia) Each square of an infinite grid is coloured in one of 5 colours, in such a way that every 5-square (Greek) cross contains one square of each colour. Show that every 1×5 rectangle also contains one square of each colour.
Note: The five colours of the Greek cross are maroon, lavender, tickeme-pink, green and neon orange.

6 (IMC) Let α be a real number such that $1 < \alpha < 2$.

(a) Show that α has a unique representation as an infinite product

$$\alpha = \prod_{i=1}^{\infty} \left(1 + \frac{1}{n_i}\right).$$

where each n_i is a positive integer satisfying

$$n_i^2 \leq n_{i+1}.$$

(b) Show that α is rational if and only if its infinite product has the following property:
For some m and all $k \geq m$,

$$n_{k+1} = n_k^2.$$

7 (Russia) An n by n square is drawn on an infinite checkerboard. Each of the n^2 cells contained in the square initially contains a token. A move consists of jumping a taken over an adjacent token (horizontally or vertically) into an empty square; the token jumped over is removed. A sequence of moves is carried out in such a way that at the end, no further moves are possible.

(a) Show that when n is even, at least $\frac{n^2}{3}$ moves have been made.

(b) Does the result still hold when n is odd?

8 (Japan) For a convex hexagon $ABCDEF$ whose side lengths are all 1, let M and m be the maximum and minimum values of the three diagonals AD, BE and CF. Find all possible values of m and M.