Singapore International Mathematics Olympiad
National Team Training

29-05-02

1 (Russia) Find all infinite bounded sequences $a_1, a_2, \ldots$ of positive integers such that for all $n > 2$,

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}, a_{n-2})}.$$

Solution

The only sequence is 2, 2, 2, ... , which clearly satisfies the given condition.
Let $g_n = \gcd(a_n, a_{n+1})$. Then g_{n+1} divides both a_{n+1} and a_{n+2}, so it divides $g_n a_{n+2} - a_{n+1} = a_n$ as well. Thus g_{n+1} divides both a_n and a_{n+1}, and it divides their greatest common divisor g_n.
Thus, the g_i form a nonincreasing sequence of positive integers and eventually equal to some positive constant g. At this point, the a_i satisfy the recursion

$$g a_n = a_{n-1} + a_{n-2}.$$

If $g = 1$, then $a_n = a_{n-1} + a_{n-2} > a_{n-1}$, so the sequence is increasing and unbounded. If $g \geq 3$, then

$$a_n = \frac{a_{n-1} + a_{n-2}}{g} < \frac{a_{n-1} + a_{n-2}}{2} \leq \max(a_{n-1}, a_{n-2}),$$

Similarly, $a_{n+1} < \max(a_{n-1}, a_n) \leq \max(a_{n-2}, a_{n-1})$, so that

$$\max(a_n, a_{n+1}) < \max(a_{n-2}, a_{n-1}).$$

Therefore the maximum values of successive pairs of terms form an infinite decreasing sequence of positive integers, a contradiction.
Thus $g = 2$ and eventually we have $2a_n = a_{n-1} + a_{n-2}$ or $a_n - a_{n-1} = -\frac{1}{2}(a_{n-1} - a_{n-2})$. This implies that $a_i - a_{i-1}$ converges to 0 and that the a_i are eventually constant as well. From $2a_n = a_{n-1} + a_{n-2}$, this constant must be 2.
Now if $a_n = a_{n+1} = 2$ for $n > 1$, then $\gcd(a_{n-1}, a_n) = \gcd(a_{n-1}, 2)$ either equals 1 or 2. Now

$$2 = a_{n+1} = \frac{a_{n-1} + a_n}{\gcd(a_{n-1}, 2)},$$

implying that either $a_{n-1} = 0$, which is impossible, or that $a_{n-1} = 2$. Thus, all the a_i equal 2, as claimed.

2 (Russia) Four natural numbers have the property that the square of the sum of any two of the numbers is divisible by the product of the **other** two. Show that at least three of the four numbers are equal.

Solution

Suppose, by the way of contradiction, four such numbers exist with no three of them equal. Select such numbers a, b, c, d such that $a+b+c+d$ is minimal. If some prime p divided both a and b, then from $a|(b+c)^2$ and $a|(b+d)^2$ we know that p divides c and d as well. Then $\frac{a}{p}, \frac{b}{p}, \frac{c}{p}, \frac{d}{p}$ is a counterexample with smaller sum. Therefore the 4 numbers are pairwise relatively prime.
Suppose that some prime $p > 2$ divided a. Because a divides each of $(b+c)^2, (c+d)^2, (d+b)^2$, we know that p divides $b+c, c+d, d+b$. Hence p divides $(b+c)+(c+d)+(d+b)$ and thus $p|(b+c+d)$. Therefore $p|(b+c+d)-(b+c)=d$, and similarly $p|c$ and $p|b$, giving a contradiction.
Thus each of a, b, c, d are powers of 2. Because they are pairwise relatively prime, three of them must equal 1, a contradiction.

3 (Iran) Let I be the incenter of triangle ABC and let AI meet the circumcircle of ABC at D. Denote the feet of the perpendiculars from I to BD and CD by E and F, respectively. If $IE + IF = \frac{AD}{2}$, calculate $\angle BAC$.

Solution

Fact: $DB = DI = DC$. In fact $\angle BDI = \angle C$ gives $\angle DIB = (\angle A + \angle B)/2$ while $\angle IBD = (\angle A + \angle B)/2$. Thus $DB = DI$ and similarly $DC = DI$.
Let $\theta = \angle BAD$. Then

$$\begin{aligned} \frac{1}{4} ID \cdot AD &= \frac{1}{2} ID \cdot (IE + IF) \\ &= \frac{1}{2} BD \cdot IE + \frac{1}{2} CD \cdot IF \\ &= \text{Area} BID + \text{Area} DIC \\ &= \frac{ID}{AD} (\text{Area} BAD + \text{Area} DAC) \\ &= \frac{1}{2} ID \cdot (AB + AC) \text{sin}\theta, \end{aligned}$$

whence $\frac{AD}{AB+AC} = 2\text{sin}\theta$.
Let X be the point on the ray $\vec{AB}$ different from A such that $DX = DA$. Because $\angle XBD = \angle DCA$ and $\angle DBX = \angle XAD = \angle DAC$, we have $\triangle XBD \sim \triangle ACD$, and $BX = AC$. Then

$$2\text{sin}\theta = \frac{AD}{AB + AC} = \frac{AD}{AB + BX} = \frac{AD}{AX} = \frac{1}{2\text{cos}\theta},$$

so that $2\text{sin}\theta\text{cos}\theta = \frac{1}{2}$, and $\angle BAC = 2\theta = 30^0$ or 150^0.

4 (Italy) Let X be a set with $|X| = n$, and let $A_1, A_2, \ldots, A_m$ be subsets of X such that

(a) $|A_i| = 3$ for $i = 1, 2, \ldots, m$.

(b) $|A_i \cap A_j| \leq 1$ for all $i \neq j$.

Prove that there exists a subset of X with at least $\lfloor \sqrt{2n} \rfloor$ elements, which does not contain A_i for $i = 1, 2, ..., m$.

Solution

Let A be a subset of X containing no A_i, and having the maximum number of elements subject to this condition. Let k be the size of A. By assumption, for each $x \in X - A$, there exists $i(x) \in 1, 2, ..., m$ such that $A_{i(x)} \subseteq A \cup \{x\}$.
Let $L_x = A \cap A_{i(x)}$, which by the previous observation must have 2 elements. Because $|A_i \cap A_j| \leq 1$ for all $i \neq j$., the L_x must be all distinct. Now there are $\binom{k}{2}$ 2-element subsets of A, so there can be at most $\binom{k}{2}$ sets L_x. Thus $n - k \leq \binom{k}{2}$ or $k^2 + k \geq 2n$. It follows that

$$k \geq \frac{1}{2}(-1 + \sqrt{1 + 8n}) > \sqrt{2n} - 1,$$

that is, $k \geq \lfloor \sqrt{2n} \rfloor$.

5 (Russia) Each square of an infinite grid is coloured in one of 5 colours, in such a way that every 5-square (Greek) cross contains one square of each colour. Show that every 1×5 rectangle also contains one square of each colour.

Note: The five colours of the Greek cross are maroon, lavender, tickeme-pink, green and neon orange.

Solution

Label the centers of the grid squares with coordinates, and suppose that square (0, 0) is coloured maroon. The Greek cross centered at (1, 1) must contain a maroon-coloured square. However, the squares (0, 1), (1, 0) and (1,1) cannot be maroon because each of the squares is in a Greek croos with (0, 0). Thus either (1, 2) or (2, 1) is maroon, WLOG, sat (1,2).

Then by a similar analysis on square (1,2) and the Greek cross centered at (2, 1), one of the squares (2, 0) and (3, 1) must be maroon. (2, 0) is in a Greek cross with (0, 0) though, so (3, 1) is maroon.

Repeating the analysis on square (2, 0) shows that (2, -1) is maroon. Spreading outward, every square of the form $(i + 2j, i - 2j)$ is maroon. Because the squares are the centers of Greek crosses that tile the plane, no other squares can be maroon; because no two of these squares are in the same 1 x 5 rectangle, no two maroon squares can be in the same 1 x 5 rectangle.

The same argument applies to all the other colours. And thus the five squares in the 1 x 5 rectangle have distinct colours, as desired.

6 (IMC) Let α be a real number such that $1 < \alpha < 2$.

(a) Show that α has a unique representation as an infinite product

$$\alpha = \prod_{i=1}^{\infty} \left(1 + \frac{1}{n_i}\right).$$

where each n_i is a positive integer satisfying

$$n_i^2 \leq n_{i+1}.$$

(b) Show that α is rational if and only if its infinite product has the following property:
For some m and all $k \geq m$,

$$n_{k+1} = n_k^2.$$

Solution

(a) We construct inductively the sequence $\{n_i\}$ and the ratios

$$\theta_k = \frac{\alpha}{\prod_{i=1}^{k} \left(1 + \frac{1}{n_i}\right)}$$

so that $\theta_k > 1$ for all k. Choose n_k to be the least n such that

$$1 + \frac{1}{n} < \theta_{k-1}$$

$(\theta_0 = \alpha)$ such that for each k,

$$1 + \frac{1}{n_k} < \theta_{k-1} \leq 1 + \frac{1}{n_k - 1}. \tag{1}$$

Hence

$$1 + \frac{1}{n_{k+1}} < \theta_k = \frac{\theta_{k-1}}{1 + \frac{1}{n_k}} \leq \frac{1 + \frac{1}{n_{k-1}}}{1 + \frac{1}{n_k}} = 1 + \frac{1}{n_k^2 - 1}.$$

Thus for each k, $n_{k+1} \geq n_k^2$. Since $n_1 \geq 2$, $n_k \to \infty$ so that $\theta_k \to 1$. Hence

$$\alpha = \prod_{i=1}^{\infty} \left(1 + \frac{1}{n_i}\right).$$

The uniqueness of the infinite product will follow from the fact that every step n_k has to be determined by (1). Indeed, if for some k we have

$$1 + \frac{1}{n_k} \geq \theta_{k-1},$$

then $\theta_k \leq 1, \theta_{k+1} < 1$ and hence $\{\theta_k\}$ does not converge to 1.
Now observe that for $M > 1$,

$$\left(1 + \frac{1}{M}\right)\left(1 + \frac{1}{M^2}\right)\left(1 + \frac{1}{M^4}\right) \ldots = 1 + \frac{1}{M} + \frac{1}{M^2} + \frac{1}{M^3} + \ldots = 1 + \frac{1}{M - 1}. \tag{2}$$

Assume that for some k we have

$$1 + \frac{1}{n_k - 1} < \theta_{k-1},$$

then we get

$$\begin{aligned}
\frac{\alpha}{\left(1+\frac{1}{n_1}\right)\left(1+\frac{1}{n_2}\right)\ldots} &= \frac{\theta_{k-1}}{\left(1+\frac{1}{n_k}\right)\left(1+\frac{1}{n_{k+1}}\right)\ldots} \\
&\geq \frac{\theta_{k-1}}{\left(1+\frac{1}{n_k}\right)\left(1+\frac{1}{n_k^2}\right)\ldots} \\
&= \frac{\theta_{k-1}}{1+\frac{1}{n_k-1}} > 1,
\end{aligned}$$

a contradiction.

(b) From (2) α is rational if its product ends in the stated way.
Conversely, suppose α is the rational number $\frac{p}{q}$. Our aim is to show that for some m,

$$\theta_{m-1} = \frac{n_m}{n_m - 1}.$$

Suppose this is not the case, so that for every m,

$$\theta_{m-1} < \frac{n_m}{n_m - 1}. \tag{3}$$

For each k we write $\theta_k = \frac{p_k}{q_k}$ as a fraction (not necessarily in the lowest terms) where $p_0 = p, q_0 = q$, and in general

$$p_k = p_{k-1} n_k, q_k = q_{k-1}(n_k + 1).$$

The numbers $p_k - q_k$ are positive integers, hence to obtain a contradiction it suffices to show that this sequence is strictly decreasing. Now,

$$\begin{aligned}
p_k - q_k - (p_{k-1} - q_{k-1}) &= n_k p_{k-1} - (n_k+1) q_{k-1} - p_{k-1} + q_{k-1} \\
&= (n_k - 1) p_{k-1} - n_k q_{k-1}
\end{aligned}$$

and this is negative since

$$\frac{p_{k-1}}{q_{k-1}} = \theta_{k-1} < \frac{n_k}{n_k - 1}$$

by inequality (3). Hence we are done.

7 (Russia) An n by n square is drawn on an infinite checkerboard. Each of the n^2 cells contained in the square initially contains a token. A move consists of jumping a taken over an adjacent token (horizontally or vertically) into an empty square; the token jumped over is removed. A sequence of moves is carried out in such a way that at the end, no further moves are possible.

(a) Show that when n is even, at least $\frac{n^2}{3}$ moves have been made.

(b) Does the result still hold when n is odd?

Solution

(a) At the end of the game no two adjacent squares contain tokens. Otherwise (because no more jumps are possible) they would have to be in an infinitely long line of tokens, which is a contradiction.
During the game, each time a token on square A jumps over another token on square B, imagine putting a 1 x 2 domino over squares A and B. At the end, every tokenless square on the checkerboard is covered by a tile, so no two uncovered squares are adjacent.
Now split the n^2 squares of the board into 2 x 2 mini boards, each containing 4 overlapping 1 x 2 tiles. At the end of the game, none of these n^2 tiles can contain 2 checkers (since no 2 checkers are adjacent at the end of the game).
Any jump removes a checker from at most three full tiles, implying that at least $\frac{n^2}{3}$ moves must have taken place.

(b) **Lemma** If an $n \times n$ square board is covered with 1×2 rectangular dominoes (possibly overlapping, and possibly with one square off the board) in such a way that no two uncovered squares are adjacent, then at least $\frac{n^2}{3}$ tiles are on the board.
Proof : Call a pair of adjacent squares on the checkerboard a **tile**. If a tile contains two squares on the border of the checkerboard, call it an **outer tile**. Otherwise, call it an **inner tile**.
Now for each domino D, consider any tile it partly covers. If this tile is partly covered by exactly m dominoes, we say that D **destroys** $\frac{1}{m}$ of the tile. Summing over all the tiles that D lies on, we find the total quantity a of outer tiles destroyed by D, and the total quantity b of inner tiles destroyed by D. We than say D scores $1.5a + b$ points.
Consider the vertical domino D consisting of the upper-left square in the chessboard and the square immediately below it. It partly destroys two horizontal tiles. One of the two squares immediately to $D's$ right must be covered, so if D destroys all of one horizontal tile, it can only destroy at most half of the other.
With this analysis, some quick checking shows that any domino scores at most 6 points. Also, it can be verified that any domino scoring 6 points (i) lies completely on the board; (ii) does not contain a corner square of the chessboard; (iii) does not overlap with any other dominoes; and (iv) does not have either length-1 edge border any other domino.
In a valid arrangement of dominoes, every tile is destroyed completely. Because there are $4(n-1)$ outer tiles and $2(n-1)(n-2)$ inner tiles, this means that a total of $1.5(4)(n-1)+2(n-1)(n-2) = 2(n^2-1)$ points are scored. Therefore, there must be at least

$$\lceil\frac{2(n^2-1)}{6}\rceil = \lceil\frac{n^2-1}{3}\rceil$$

dominoes.

Suppose, by way of contradiction, that we have **exactly** $\frac{n^2-1}{3}$ dominoes. For this to be an integer, n must not be divisible by 3. In addition, the restrictions described earlier must hold for every domino.
Suppose we have any horizontal domino not at the bottom of the chessboard. One of the two squares directly below it must be covered. To satisfy our restrictions, this square must be covered by a horizontal domino (not a vertical one). Thus we can find a chain of horizontal dominoes stretching to the bottom of the board. Similarly, we can follow this chain to the top of the board.
Likewise, if there is any vertical domino then some chain of vertical dominoes streches across the board. However, we cannot have both a horizontal and a vertical chain that do not overlap, so all the dominoes must have the same orientation: WLOG, suppose they are all horizontal.
To cover the tiles in any given row while satisfying the restrictions, we must alternate between blank squares and horizontal dominoes. In the top row, because no dominoes contain corner squares, we must start and end with blank squares. Thus $n \equiv 1 \pmod 3$. Then in the second row, we must start with a horizontal domino (to cover the top-left vertical tiles). After alternating between dominoes and blank squares, the end of the row will contain two blank squares, a contradiction. Thus it is impossible to cover the chessboard with exactly $\frac{n^2-1}{3}$ dominoes, and indeed at least $\frac{n^2}{3}$ dominoes are needed.

8 (Japan) For a convex hexagon $ABCDEF$ whose side lengths are all 1, let M and m be the maximum and minimum values of the three diagonals AD, BE and CF. Find all possible values of m and M.

Solution

We claim that $\sqrt{3} \le M \le 3$ and $1 \le m \le 2$.
First, we show all such values are attainable. Continuously transform $ABCDEF$ from an equilateral triangle ACE of side length 2, into a regular hexagon of side length 1, and finally into a segment of length 3 (for instance, by enlarging the diagonal AD of the regular hexagon while bringing B, C, E, F closer to line AD.) Then M continuously varies from $\sqrt{3}$ to 2 to 3. Similarly, by continuously transforming $ABCDEF$ from a $1x2$ rectangle into a regular hexagon, we can make m vary continuously from 1 to 2.
Now we prove no other values are attainable. First, we have $AD \le AB+BC+CD = 3$ and similarly, $BE, CF \le 3$ so that $M \le 3$.
Next suppose, by way of contradiction, that $m < 1$ and assume WLOG that $AD < 1$. Because $AD < AB = BC = CD = 1$,

$$\angle DCA < \angle DAC, \angle ABD < \angle ADB,$$

$$\angle CBD = \angle CDB, \angle BCA = \angle BAC.$$

Therefore,

$$\begin{aligned} \angle CDA + \angle BAD &= \angle CDB + \angle BDA + \angle BAC + \angle CAD \\ &> \angle CBD + \angle DBA + \angle BCA + \angle ACD \\ &= \angle CBA + \angle BCA. \end{aligned}$$

Consequently, $\angle CDA + \angle BAD > 180^0$ and likewise $\angle EDA + \angle FAD > 180^0$. Then

$$\angle CDE + \angle BAF = \angle CDA + \angle EDA + \angle BAD + \angle FAD > 360^0,$$

which is impossible since $ABCDEF$ is convex. Hence $m \ge 1$.
Next we demonstrate that $M \ge \sqrt{3}$ and $m \le 2$. Because the sum of the six interior angles on $ABCDEF$ is 720^0, some pair of adjacent angles has sum greater than or equal to 240^0 and some pair has sum less than or equal to 240^0. Thus it suffices to prove that $CF \ge \sqrt{3}$ when $\angle A + \angle B \ge 240^0$, and that $CF \le 2$ when $\angle A + \angle B \le 240^0$. By the law of cosines,

$$CF^2 = BC^2 + BF^2 - 2BC \cdot BF \cos \angle FBC.$$

Thus if we fix A, B, F and decrease $\angle ABC$, we decrease $\angle FBC$ and CF. Similarly, by fixing A, B, C and decreasing $\angle BAF$, we decrease CF. Therefore, it suffices to prove that $CF \le 2$ when $\angle A + \angle B = 240^0$.
Now suppose that $\angle A + \angle B$ does equal 240^0. Let lines AF and BC intersect at P, and set $x = PA$ and $y = PB$. Because $\angle A + \angle B = 240^0$, $\angle P = 60^0$. Then applying the law of cosines to triangles PAB and PCF yields

$$1 = AB^2 = x^2 + y^2 - xy$$

and

$$CF^2 = (x+1)^2 + (y+1)^2 - (x+1)(y+1) = 2 + x + y.$$

Therefore, we need only find the possible values of $x + y$ given that $x^2 + y^2 - xy = 1$ and $x, y \ge 0$. These conditions imply that

$$(x+y)^2 + 3(x-y)^2 = 4, x + y \ge 0,$$

and $|x-y| \le x+y$. Hence,

$$\begin{aligned} 1 &= \frac{1}{4}(x+y)^2 + \frac{3}{4}(x-y)^2 \\ &\le (x+y)^2 \le (x+y)^2 + 3(x-y)^2 = 4, \end{aligned}$$

so $1 \le x+y \le 2$ and $\sqrt{3} \le CF \le 2$. This completes the proof.